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COMPLETENESS IN ZARISKI GROUPS

BY

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ABSTRACT

Zariski groups are $\aleph_0$-stable groups with an axiomatically given Zariski topology and thus abstract generalizations of algebraic groups. A large part of algebraic geometry can be developed for Zariski groups. As a main result, any simple smooth Zariski group interprets an algebraically closed field, hence is almost an algebraic group over an algebraically closed field.

Introduction

Model theory came naturally across the notion of $\aleph_0$-stable groups of finite Morley rank. These are groups where a finite dimension is assigned to all definable sets that behaves much as dimension in algebraic geometry. In fact, these groups share many properties with algebraic groups. It was even conjectured that they were essentially algebraic groups. Removing trivial obstacles, Cherlin's conjecture states that a simple $\aleph_0$-stable group (of finite rank) is an algebraic group over an algebraically closed field. The theory is now well developed: see [BN] for an algebraic and [P2] for a more model theoretic introduction to the subject.

The main results about $\aleph_0$-stable groups of finite Morley rank are the following: If an $\aleph_0$-stable group of finite Morley rank has a definable, connected, solvable non-nilpotent subgroup, then an algebraically closed field is interpretable (Zil'ber). If the group is simple, it is interpretable in the field possibly equipped with some extra structure coming from the group (Hrushovski). Hence the main problems are to eliminate the so-called bad groups, where no connected, solvable,

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non-nilpotent subgroups are definable, and to show that the group is algebraic over the field it interprets.

As long as the general Cherlin conjecture is still unsolved, it is natural to consider weaker forms. One possibility is motivated by Hrushovski's and Zil'ber's work on strongly minimal sets and Zil'ber's conjecture, which is a similar problem. The main part of Zil'ber's conjecture asserted that a non-locally modular strongly minimal set was an algebraic curve. Hrushovski constructed counter-examples to this, but Hrushovski and Zil'ber succeeded in proving the conjecture for special strongly minimal sets, so-called Zariski geometries. These are structures equipped with Noetherian topologies as abstract Zariski topologies. More precisely, their result characterizes the Zariski topology of smooth algebraic curves over algebraically closed fields.

In the light of their work, it seems natural to consider "Zariski groups" defined in this article: $\aleph_0$-stable groups with an axiomatically given Zariski topology.

The main interest of Cherlin's conjecture, at least from an algebraic point of view, is the hope to get an abstract characterization of algebraic groups, not mentioning fields and varieties. While a positive solution of the general conjecture would characterize the constructible structure of algebraic groups, a solution for Zariski groups would provide a characterization of the Zariski topology of algebraic groups.

This article gives an approach to Cherlin's conjecture for Zariski groups. In particular I show that there are no bad smooth Zariski groups, hence that any simple smooth Zariski group interprets an algebraically closed field. Most probably, this result follows also from Hrushovski's and Zil'ber's work [HZ1], [HZ2]. The problem is to show that there is a strongly minimal subset of a smooth Zariski group satisfying the dimension formula. Anyhow, I hope my proof is of interest because my methods are more elementary and might be more easily understood.

1. Zariski groups

1.1 SOME TOPOLOGICAL PREREQUISITES. Let T be a Noetherian topological space, i.e. a space without infinite strictly descending chains of closed sets. A subset of T is **irreducible** if it is non-empty and not the union of two proper relatively closed subsets. An **irreducible component** of a set is a maximal irreducible subset. Any subset of T is the union of its finitely many irreducible components.

The (topological) **dimension** of an irreducible subset X of T is defined by induction: $\dim \emptyset := -1$ and $\dim X := \sup\{\dim Y | Y \subseteq X, Y$ irreducible and not dense in $X\} + 1$. The dimension of an arbitrary subset is the maximum of the dimensions of its irreducible components. In the case it is finite, dimension is just the maximal length of a chain of relatively closed irreducible subsets. A **hypersurface** of X is a relatively closed irreducible subset H such that $\dim H + 1 = \dim X$.

A **constructible** set is a Boolean combination of closed sets. Any constructible set contains a dense open subset of its closure, in other words $\overline{\overline{Q} \setminus Q}$ is a proper subset of $\overline{Q}$. In particular, $\dim \overline{\overline{Q} \setminus Q} < \dim \overline{Q}$. A set cannot be the disjoint union of two constructible dense subsets.

Definable always means definable with parameters, unless otherwise specified. Moreover, $\subset$ and $\supset$ are always proper inclusions.

1.2 THE COMBINATORIAL PART OF THE AXIOMS. Let G be an infinite group, where μ and ι denote the maps $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ respectively, together with a Noetherian topology τ_n on G^n for each $n \in \omega$. Such a group G is a **Zariski group** iff the three groups of axioms A, B and C below are satisfied.

Definition 1.1: A map $\phi : G^k \to G^m$ is a **morphism** iff all maps $\phi \times \mathrm{id}_{G^n} : G^{k+n} \to G^{m+n}$ are continuous.

Axioms A:

(1) G is irreducible in the topology τ_1.

(2) The diagonal $\Delta(G) = \{(g, g) | g \in G\}$ is closed in τ_2.

(3) The following maps are morphisms:

 the projections $G^2 \to G$;

 the diagonal map $G \to G^2$, $g \mapsto (g, g)$;

 the permutation $G^2 \to G^2$, $(g_1, g_2) \mapsto (g_2, g_1)$;

 the constant maps $G \to G$, $g \mapsto a$, for all $a \in G$.

(4) All projections $G^{n+1} \to G^n$ map constructible sets onto constructible ones.

(5) The multiplication μ and the inverse map ι are morphisms.

Remarks: The compatibility, i.e. Axiom A3, relates the different topologies τ_n and provides some symmetry and homogeneity properties. For example, the induced topology on $\{a\} \times G$ is naturally homeomorphic to τ_1. In particular, all projections, permutations, diagonal maps and constant maps with domain G^n are morphisms.

Axiom A1 implies that the group is connected, i.e. with no proper closed subgroup of finite index.

Axiom A4 is Chevalley's theorem in the case of algebraic groups and in general corresponds to quantifier elimination in an appropriate language.

Finally, Axiom A5 reflects the condition in the definition of algebraic groups that μ and ι are morphisms in the sense of algebraic geometry.

Notation: If X is a subset of G^{n+k} and $\bar{a} \in G^n$, then $X_{\bar{a}}$ stands for the set $\pi^{-1}(\bar{a}) \cap X$ for some fixed projection $\pi : G^{n+k} \to G^n$ and is called the (π-) **fiber** of X over $\bar{a}$.

LEMMA 1.2:

(a) *Cartesian products of closed sets are closed. Points in G^n are closed. Fibers $F_{\bar{a}}$ of a closed set F are closed.*

(b) *The graph Γ_f of a morphism $f : G^k \to G^m$ is closed.*

Proof: (a) Let $F_i \subseteq G^{n_i}$ be closed. Then $F_i \times G^k$ is closed by continuity of the projections, and $F_1 \times F_2 = (F_1 \times G^{n_2}) \cap (G^{n_1} \times F_2)$. Next consider the morphism $f : G \to G^2$, $g \mapsto (g, a)$. Then $\{a\} = f^{-1}[\Delta(G)]$ is closed, whence $\{(a_1, \ldots, a_n)\} = \{a_1\} \times \cdots \times \{a_n\}$ also. Now $F_{\bar{a}} = \pi^{-1}(\bar{a}) \cap F$ is closed because π is continuous.

(b) $\Gamma_f = (f \times \mathrm{id}_{G^m})^{-1}[\Delta(G^m)]$ is closed by Axiom A1 and (a) because $\Delta(G^m)$ is homeomorphic to $\Delta(G)^m$. ∎

1.3 ZARISKI GROUPS AS FIRST-ORDER STRUCTURES. A first-order language for G is any relational language (with equality) such that the interpretations of the quantifier-free positive formulae with parameters are exactly the closed sets. (It is convenient, but not necessary, to add a function symbol for multiplication.) By Axiom A3 and Lemma 1.2 (a), a possible language is given by all closed sets, but in some cases it might be more natural to consider smaller languages. Now Axiom A4 says that G as a first-order structure admits elimination of quantifiers. In other words, any definable set is constructible.

Axioms B:

(1) G admits a countable language.

(2) There are no arbitrarily long uniformly defined descending chains of closed sets, i.e. there are no closed sets $F^{(i)}$ and tuples of parameters $\bar{a}_{ij}$ such that $F^{(0)}_{\bar{a}_{0k}} \supset F^{(1)}_{\bar{a}_{1k}} \supset \cdots \supset F^{(k)}_{\bar{a}_{kk}}$ for all $k \in \omega$.

Remark: Axiom B2 guarantees that an elementary extension of G is still a Zariski group. It does not depend on the choice of the language. Axiom B1 allows one to show that G is an $\aleph_0$-stable structure.

Fix for the sequel a countable first-order language $\mathcal{L}$ for G. The closed sets named in the language are called **basic closed sets**. Let G^* be an $\mathcal{L}$-structure that is elementarily equivalent to G. There is a natural notion of closed subsets of G^{*n}, namely the sets defined by quantifier-free positive formulae with parameters in G^*.

PROPOSITION 1.3: *The naturally defined closed sets on G^{*n} define a Noetherian topology. G^* together with the family of these topologies satisfies Axioms* A.

Proof: The class of closed sets is clearly stable under finite unions and finite intersections. DCC on closed sets follows easily by compactness from Axiom B2, thus we have a Noetherian topology.

Reducibility of a closed set F, say $F = F'_{\bar{a}} \cup F''_{\bar{b}}$ with basic closed sets F', F'', is first-order expressible. Hence the irreducibility is part of the first-order theory.

The diagonal, being a basic closed set, remains closed in G^*.

Morphisms are definable maps by Lemma 1.2 (b). They are in fact uniformly continuous in the following sense: if ϕ is a morphism and F a closed set, then $\phi^{-1}[F_{\bar{a}}] = \phi^{-1}[F]_{\bar{a}}$ for all $\bar{a}$. Hence continuity is first-order expressible for each basic closed set.

Quantifier elimination clearly remains true for constructible sets of the form $Q_{\bar{a}}$ where Q is $\emptyset$-definable. But any constructible set in G^* may be written in this form. (For example $F_{\bar{a}} \cap F'_{\bar{b}}$ is homeomorphic to $(F \times F')_{\bar{a}\frown\bar{b}}$.) ∎

Axioms C: For some $\aleph_0$-saturated extension G^* of G,

(1) $\dim G^*$ is finite;

(2) every closed irreducible infinite set F in G^{*n} (for all n) has infinitely many hypersurfaces.

Remark: Axiom C2 implies that Morley rank equals the topological dimension. Then Axiom C1 just states that the structure is of finite Morley rank.

THEOREM 1.4: *Let G be a Zariski group in a fixed first-order language.*

(a) *G is an $\aleph_0$-stable structure of finite Morley rank.*

(b) *Any elementarily equivalent structure G^* is also a Zariski group.*

First recall that the Cantor rank RC of a definable set Q in a structure is defined as follows:

- $\mathrm{RC}(Q) \geq 0 \iff Q \neq \emptyset$;
- $\mathrm{RC}(Q) \geq \alpha + 1 \iff$ for any n there are n disjoint definable $Q_i \subseteq Q$ with $\mathrm{RC}(Q_i) \geq \alpha$;

- $\mathrm{RC}(Q) \geq \beta \iff \mathrm{RC}(Q) \geq \alpha$ for all $\alpha < \beta$ for a limit ordinal β;
- Finally, let $\mathrm{RC}(Q) = \sup\{\alpha | \mathrm{RC}(Q) \geq \alpha\}$.

Then $\mathrm{RC}(Q_1 \cup Q_2) = \max\{\mathrm{RC}(Q_1), \mathrm{RC}(Q_2)\}$; $Q_1 \subseteq Q_2$ implies $\mathrm{RC}(Q_1) \leq \mathrm{RC}(Q_2)$ and $\mathrm{RC}(Q) = 0 \iff Q$ is finite.

LEMMA 1.5: *The Cantor rank in G is bounded by the dimension.*

Proof: Induction on Cantor rank: If $\mathrm{RC}(Q) = 0$, then Q is finite, hence $\dim Q = 0$. Let $\mathrm{RC}(Q) = n+1$. One of the irreducible components of Q, say Q', must have rank $n+1$. By definition of Cantor rank, there are two disjoint constructible subsets Q_1, Q_2 of Q' of rank n, which one may suppose to be irreducible by taking appropriate components. Q_1, Q_2 being disjoint, there is one, say Q_1, such that $\overline{Q_1} \neq \overline{Q'}$. It follows that $\dim Q \geq \dim Q' > \dim(\overline{Q_1} \cap Q') \geq \dim Q_1$ and by induction $\dim Q_1 \geq \mathrm{RC}(Q_1) = n$. ∎

Because Morley rank equals Cantor rank in an $\aleph_0$-saturated structure, the preceding lemma together with Axiom C1 yields immediately that G is a totally transcendental structure of finite Morley rank, hence $\aleph_0$-stable by Axiom B1 and proposition 17.12 of [P1].

LEMMA 1.6: *Let G be any structure equipped with a family of Noetherian topologies satisfying Axioms* A1 *to* A4. *Then the following are equivalent:*

(a) $\mathrm{RC}(Q) = \dim Q$ *for all constructible sets Q.*

(b) $\dim Q = \dim \overline{Q}$ *for all constructible sets Q.*

(c) *Axiom* C2.

Note that if Axiom C2 holds for a closed irreducible set F and if F' is a proper closed subset of F, then it is possible to find infinitely many hypersurfaces H of F such that $H \setminus F'$ is dense in H: by irreducibility of H, either $H \cap F'$ or $H \setminus F'$ is dense in H. But in the first case, $H = \overline{H \cap F'} \subseteq F'$, hence H is an irreducible component of F', which is only possible for finitely many H.

Proof: The case of dimension 0 is always clear because $\dim Q = 0 \iff \dim \overline{Q} = 0 \iff \mathrm{RC}(Q) = 0 \iff Q$ is finite.

(b)⇒(c): Assume $H_1, \ldots, H_k$ are all the hypersurfaces of F. Then $\overline{F \setminus (H_1 \cup \cdots \cup H_k)} = F$, hence $\dim[F \setminus (H_1 \cup \cdots \cup H_k)] = \dim F$ and there is a relatively closed irreducible $H \subset F \setminus (H_1 \cup \cdots \cup H_k)$ of dimension $\dim F - 1$. Thus $\overline{H}$ is a hypersurface of F. But clearly $\overline{H} \neq H_i$: contradiction.

(c)⇒(b): Induction on $\dim \overline{Q}$. It is sufficient to consider irreducible Q. According to the above remark, there is a hypersurface H of $\overline{Q}$ such that $H \cap Q$

is dense in H. Then $\dim Q > \dim(H \cap Q)$, and by induction $\dim(H \cap Q) = \dim H = \dim \overline{Q} - 1$.

(a)$\Rightarrow$(b): $\dim \overline{Q} = \mathrm{RC}(\overline{Q}) = \max\{\mathrm{RC}(Q), \mathrm{RC}(\overline{Q} \setminus Q)\} = \max\{\dim Q, \dim(\overline{Q} \setminus Q)\} = \dim Q$, because $\dim \overline{Q} \setminus Q < \dim \overline{Q}$.

(b),(c)$\Rightarrow$(a): Again induction on $\dim \overline{Q}$. It is sufficient to consider irreducible Q. Let $\dim Q = n+1$. Choose infinitely many different hypersurfaces H_i of $\overline{Q}$ such that $\overline{H_i \cap Q} = H_i$ (see remark above). Let $Q_0 := Q \cap H_0$ and inductively $Q_{k+1} := Q \cap H_{k+1} \setminus (H_0 \cup \cdots \cup H_k)$. Then $\overline{Q_i} = H_i$, the Q_i are disjoint, thus $\mathrm{RC}(Q) \geq \mathrm{RC}(Q_i) + 1$ and by induction $\mathrm{RC}(Q_i) = \dim Q_i = \dim \overline{Q_i} = n$. ∎

LEMMA 1.7: *Suppose G, G^* are two $\aleph_0$-saturated elementary substructures of a Zariski group. Then the dimension of a $(G \cap G^*)$-definable closed set F is the same in both geometries G and G^*.*

Proof: Suppose $F^{(0)}_{\bar{a}_0} \subset \cdots \subset F^{(k)}_{\bar{a}_k} \subseteq F$ is a chain of closed irreducible sets of maximal length in G. Realize $\mathrm{tp}(\bar{a}_0, \ldots, \bar{a}_k)$ in G^* by $(\bar{b}_0, \ldots, \bar{b}_k)$. Then $F^{(0)}_{\bar{b}_0} \subset \cdots \subset F^{(k)}_{\bar{b}_k} \subseteq F$ is a chain of closed irreducible sets in G', whence $\dim_G F \leq \dim_{G^*} F$. Equality follows by symmetry of the situation. ∎

In particular, if Morley rank equals the dimension in one $\aleph_0$-saturated group, then the equality holds in all $\aleph_0$-saturated models.

Now it is easy to prove part (b) of Theorem 1.4: G^* satisfies Axioms A by Proposition 1.3 and Axioms B trivially. By Lemma 1.6, Axiom C2 is equivalent to the fact that dimension equals Morley rank in some $\aleph_0$-saturated elementary extension. But dimensions in $\aleph_0$-saturated models coincide (Lemma 1.7), hence Axioms C are true for G^*.

Recall that Morley rank is definable and additive in $\aleph_0$-stable groups of finite rank ([P2] corollary 2.14). It is not difficult to see that if Morley rank is definable then it equals Cantor rank in all models, and moreover, if Morley rank equals dimension in an $\aleph_0$-saturated model then this equality holds in any model (cf. [P2] p. 50).

Examples:

- Any algebraic group over an algebraically closed field is a Zariski group, the topologies being the Zariski topologies. (The axioms are easily verified once algebraic morphisms are shown to be morphisms in the sense of Definition 1.1. See [Hu] for details.)
- Any $\aleph_0$-stable one-based group of finite rank is a Zariski group, closed sets being the cosets of definable subgroups (see [HP]).

2. Varieties and morphisms

2.1 VARIETIES. Fix a Zariski group G and let V be an imaginary set in G, i.e. a definable subset W of some G^n divided by a definable equivalence relation E. Then there is a natural family of Noetherian topologies on the products $V \times G^n$: first take the induced topology on $W \times G^n$ and then its quotient topology under the canonical surjection $p_n : W \times G^n \to W/E \times G^n$.

Definition 2.1: An imaginary set $V = W/E$ equipped with these natural topologies on the products $V \times G^n$ is called a **variety** iff E is closed in $W \times W$.

With this general definition, varieties are not necessarily well behaved. For a good general theory, one has to restrict carefully the equivalence relations (see [J] chapter 3). But here we will only be concerned with subgroups and quotient groups where the definition above is sufficient.

Note that G is itself a variety, hence properties of varieties or definitions for varieties apply to G, too. If $V_1 = W_1/E_1$ and $V_2 = W_2/E_2$ are varieties, then $E_1 \times E_2$ is a closed set and defines an equivalence relation on $W_1 \times W_2$. Hence the product $V_1 \times V_2 = (W_1 \times W_2)/(E_1 \times E_2)$ is a variety.

It is obvious from the definition that points in a variety are closed as well as the diagonal.

2.2 MORPHISMS. Definition 1.1 immediately extends to varieties:

Definition 2.2: Let V_1, V_2 be two varieties. A **morphism** is a map $f : V_1 \to V_2$ such that $f \times \mathrm{id}_{G^n} : V_1 \times G^n \to V_2 \times G^n$ is continuous for every $n \in \omega$.

As in Lemma 1.2(b), morphisms are definable maps with closed graph. In [HZ1], morphisms are defined to be maps with closed graph. Only in special situations is this sufficient to show continuity (cf. Proposition 2.10).

The identity map, constant maps, projections and diagonal maps are examples of morphisms. Inclusion maps are morphisms. If E is a closed equivalence relation on V, then the canonical surjection $p : V \to V/E$ is a morphism.

LEMMA 2.3: *If $f : V_1 \to V_2$ is a morphism, then for every variety V_3, the map $f \times \mathrm{id}_{V_3} : V_1 \times V_3 \to V_2 \times V_3$ is continuous.*

Proof: Let $V_i = W_i/E_i$ where $W_i \subseteq G^{n_i}$ is definable, E_i a closed equivalence relation on W_i and $p_i : W_i \to V_i$ the canonical surjection. Consider the following commutative diagram:

$$\begin{array}{ccc}
V_1 \times W_3 & \xrightarrow{f \times \mathrm{id}_{W_3}} & V_2 \times W_3 \\
\Big\downarrow{\scriptstyle \mathrm{id}_{V_1} \times p_3} & & \Big\downarrow{\scriptstyle \mathrm{id}_{V_2} \times p_3} \\
V_1 \times V_3 & \xrightarrow{f \times \mathrm{id}_{V_3}} & V_2 \times V_3
\end{array}$$

Let $F \subseteq V_2 \times V_3$ be closed. Then $(f \times \mathrm{id}_{W_3})^{-1} \circ (\mathrm{id}_{V_2} \times p_3)^{-1}[F]$ is E_3-saturated and relatively closed in $V_1 \times W_3$, hence $(f \times \mathrm{id}_{V_3})^{-1}[F] = (\mathrm{id}_{V_1} \times p_3) \circ (f \times \mathrm{id}_{W_3})^{-1} \circ (\mathrm{id}_{V_2} \times p_3)^{-1}[F]$ is closed in $V_1 \times V_3$. ∎

PROPOSITION 2.4: *Products, pairs and compositions of morphisms are morphisms. If $f : V_1 \times V_2 \to V$ is a morphism, then for all $c \in V_1$, the map $f_c : V_2 \to V$, $v \mapsto f(c, v)$ is a morphism.*

Proof: Composition: $(g \circ f) \times \mathrm{id}_{G^k} = (g \times \mathrm{id}_{G^k}) \circ (f \times \mathrm{id}_{G^k})$ is continuous.

If $f_i : V_i \to W_i$ are morphisms, the product $f_1 \times f_2 : V_1 \times V_2 \to W_1 \times W_2$ is a morphism because $f_1 \times f_2 \times \mathrm{id}_{G^k} = (f_1 \times \mathrm{id}_{W_2} \times \mathrm{id}_{G^k}) \circ (\mathrm{id}_{V_1} \times f_2 \times \mathrm{id}_{G^k}) = (f_1 \times \mathrm{id}_{W_2 \times G^k}) \circ \sigma (f_2 \times \mathrm{id}_{V_1 \times G^k}) \tau$ for some permutations σ, τ, hence $f_1 \times f_2 \times \mathrm{id}_{G^k}$ is continuous.

If $V_1 = V_2$, the pair $(f_1, f_2) : V_1 \to W_1 \times W_2$ is a morphism because $(f_1, f_2) = (f_1 \times f_2) \circ \delta$ where $\delta : V_1 \to V_1 \times V_1$ is the diagonal morphism $v \mapsto (v, v)$.

Finally, let $\tilde{c}$ be the constant morphism $V_1 \to V_1$, $v \to c$. Then $f_c = f \circ (\tilde{c} \times \mathrm{id}_{V_2})$. ∎

LEMMA 2.5: *Let V, W and V/E be varieties where E is a definable equivalence relation on V. If $\phi : V \to W$ is a morphism that is constant on E-classes, then factoring through V/E provides a morphism $\bar{\phi} : V/E \to W$.*

Proof: Consider the following commutative diagram:

$$\begin{array}{ccc}
V \times G^n & \xrightarrow{p \times \mathrm{id}_{G^n}} & V/E \times G^n \\
& {\scriptstyle \phi \times \mathrm{id}_{G^n}} \searrow \qquad \swarrow {\scriptstyle \bar{\phi} \times \mathrm{id}_{G^n}} & \\
& W \times G^n &
\end{array}$$

Take $F \subseteq W \times G^n$ closed. Then $(\phi \times \mathrm{id}_{G^n})^{-1}[F]$ is closed and E-saturated, hence $(\bar{\phi} \times \mathrm{id}_{G^n})^{-1}[F] = (p \times \mathrm{id}_{G^n})[(\phi \times \mathrm{id}_{G^n})^{-1}[F]]$ is closed. ∎

2.3 SMOOTHNESS. A Noetherian space T satisfies the **dimension formula** if for all closed irreducible subsets F_1, F_2 and each irreducible component X of $F_1 \cap F_2$ the following holds:

$$\dim X + \dim T \geq \dim F_1 + \dim F_2.$$

Because irreducible components are non-empty by definition, this inequality holds in particular, if $F_1 \cap F_2 = \emptyset$.

Definition 2.6: A variety V is **smooth** iff the dimension formula holds in all $V \times G^n$ $(n \in \omega)$.

Remark: The terminology comes from the fact that a smooth variety in the algebraic sense satisfies the dimension formula. The converse does not hold. In [HZ1], smoothness is part of the axiom for Zariski geometries.

Let V, W be varieties, $\pi : V \times W \to V$ the projection and $Q \subseteq V \times W$ an irreducible definable set. Define $\pi[Q, \geq k]$ to be the set of all $\bar{a}$ such that the π-fiber $Q_{\bar{a}}$ over $\bar{a}$ is of dimension at least k. The dimension is **definable** in $V \times W$ iff all the sets $\pi[Q, \geq k]$ are definable. The dimension is **semi-continuous** iff the sets $\pi[F, \geq k]$ are closed in $\pi[F]$ for closed F.

If the dimension is definable and $\pi[Q]$ is irreducible, then there is exactly one k such that $\pi[Q, k] := \pi[Q, \geq k] \setminus \pi[Q, \geq k+1]$ is dense in $\pi[Q]$. It is called the **π-generic fiber dimension** of Q, denoted by $\pi\text{-gdim}\, Q$. Semi-continuity (for all closed F) is equivalent to the fact that $\pi\text{-gdim}\, F$ is the minimal dimension of the (non-empty) π-fibers of F.

Finally, dimension is **additive** in $V \times W$ iff the equation $\pi\text{-gdim}\, Q = \dim Q - \dim \pi[Q]$ holds for all irreducible Q and all π. In particular, this is the case if V and W are powers of the group G.

Definition 2.7: A variety V is **rich** if for any $a \in V$, the intersection of all hypersurfaces of V containing a is finite.

LEMMA 2.8: *A simple smooth Zariski group G is rich, as are its powers G^n.*

Proof: It suffices to verify richness for $(e, \ldots, e)$, because any point in G^k can be translated onto $(e, \ldots, e)$ by an isomorphism.

Define $G^{\mathrm{h}} := \bigcap\{H \mid H \text{ a hypersurface of } G \text{ and } e \in H\}$. By Noetherianity, G^{h} is definable. It is easy to see that G^{h} is a proper normal subgroup of G, thus $G^{\mathrm{h}} = \{e\}$.

If H is a hypersurface in G, any set $G^m \times H \times G^{k-m-1}$ is a hypersurface in G^k by additivity of dimension. Hence the intersection of all hypersurfaces in G^k containing $(e, \ldots, e)$ equals $G^{\mathrm{h}} \times \cdots \times G^{\mathrm{h}} = \{(e, \ldots, e)\}$. ∎

The smoothness condition implies many structural properties as for example in the following two propositions:

PROPOSITION 2.9: *Suppose V, W are varieties such that $V \times W$ is smooth, V is rich, and dimension is additive in $V \times W$. Let $\pi : V \times W \to V$ be the projection, $F \subseteq V \times W$ a closed irreducible set and $a \in \pi[F]$. Then $\dim H \geq \dim F - \dim \pi[F]$ for each irreducible component H of $\pi^{-1}(a) \cap F$. In particular, dimension is semi-continuous.*

Proof: Construct subsets X_i of $\pi[F]$ by induction such that:

X_i is irreducible, closed in $\pi[F]$ and contains a;

$\dim X_i = \dim \pi[F] - i$;

$\dim H \geq \dim F - i$ for each irreducible component H of $\pi^{-1}[X_i] \cap F$.

Let $X_0 := \pi[F]$. The conditions are obviously satisfied. Suppose that X_i is constructed for some $i < \dim \pi[F]$. In particular, X_i is infinite. By richness, there is a hypersurface of V containing a but not all X_i. Let X_{i+1} be an irreducible component of $X_i \cap H_i$ containing a. Then X_{i+1} is a proper closed subset of X_i, whence $\dim X_{i+1} < \dim X_i$. On the other hand, $\dim X_{i+1} \geq \dim X_i + \dim H_i - \dim V = \dim X_i - 1$ by smoothness of V, which follows immediately from the smoothness of $V \times W$ by additivity of the dimension. Hence $\dim X_{i+1} = \dim X_i - 1 = \dim F - i - 1$.

Now $\pi^{-1}[X_{i+1}] \cap F$ is a union of irreducible components of $(H_i \times W) \cap \pi^{-1}[X_i]$. Again by smoothness, each irreducible component H of $\pi^{-1}[X_{i+1}] \cap F$ has at least dimension

$$\dim(H_i \times W) + \dim(X_i \times W) - \dim(V \times W) =$$
$$\dim V - 1 + \dim W + \dim \pi[F] - i - \dim V - \dim W = \dim \pi[F] - i - 1.$$

Hence $\dim X_{\dim \pi[F]} = 0$, which means $X_{\dim \pi[F]} = \{a\}$ and finishes the proof. ∎

PROPOSITION 2.10: *Suppose V, W are varieties, $V \times W$ is smooth and $g : V \to W$ a map whose graph Γ_g is closed irreducible. Then g is continuous. (This is lemma 5.5 of [HZ1].)*

Proof: Let H be closed irreducible in W and F an irreducible component of $\overline{g^{-1}[H]}$. There is closed $F_1 \subset F$ such that $F \setminus F_1 \subseteq g^{-1}[H]$, namely $F_1 = \overline{F \setminus g^{-1}[H]}$. Then $(F \times W) \cap \Gamma_g \subseteq [(F_1 \times W) \cap \Gamma_g] \cup [(V \times H) \cap \Gamma_g]$.

Now $\dim(F_1 \times W) \cap \Gamma_g = \dim F_1 < \dim F$, but the dimension formula yields $\dim X \geq \dim(F \times W) + \dim \Gamma_g - \dim(V \times W) = \dim F$ for each irreducible component X of $(F \times W) \cap \Gamma_g$. It follows that $(F \times W) \cap \Gamma_g \subseteq (V \times H) \cap \Gamma_g$, that is $F_1 = \emptyset$ and $g^{-1}[H]$ is closed. ∎

COROLLARY 2.11: *Suppose V and W are irreducible varieties such that $V \times W$ is smooth.*

(a) *$g : V \to W$ is a morphism iff Γ_g is closed irreducible.*

(b) *A bijective morphism is an isomorphism.*

Proof: (a) If Γ_g is closed irreducible, $\Gamma_{g \times \mathrm{id}_{G^n}} = \Gamma_g \times \Delta(G^n)$ is closed irreducible, too. All $(V \times G^n) \times (W \times G^n)$ are smooth by definition, hence all $g \times \mathrm{id}_{G^n}$ are continuous. The other direction is clear.

(b) is immediate from (a). ∎

2.4 COMPLETENESS.

Definition 2.12: Let G be a Zariski group. A variety C is **complete** iff all the projections $C \times G^n \to G^n$ map closed sets onto closed sets $(n \geq 1)$.

If G is the base field of an algebraic variety, this definition corresponds exactly to that of a complete variety in the algebraic sense (Lemma 2.13). The basic properties of complete algebraic varieties (see [Hu] 6.1) remain true (Lemma 2.14).

LEMMA 2.13: *If C is a complete variety and V is a variety, then the projection $\pi : C \times V \to V$ is a closed map.*

Proof: Let $V = W/E$ and $p : W \to V$ be the canonical surjection. The diagram

$$\begin{array}{ccccc}
C \times G^n & \supseteq & C \times W & \xrightarrow{\mathrm{id}_C \times p} & C \times V \\
\Big\downarrow \pi & & \Big\downarrow \pi \restriction_{C \times W} & & \Big\downarrow \pi' \\
G^n & \supseteq & W & \xrightarrow{\quad p \quad} & V
\end{array}$$

is commutative, hence if $F \subseteq C \times V$ is closed, then $\pi\left[\overline{(\mathrm{id}_C \times p)^{-1}[F]}\right]$ is E-saturated and relatively closed in W, whence $\pi'[F] = p\left(W \cap \pi\left[\overline{(\mathrm{id}_C \times p)^{-1}[F]}\right]\right)$ is closed, too. ∎

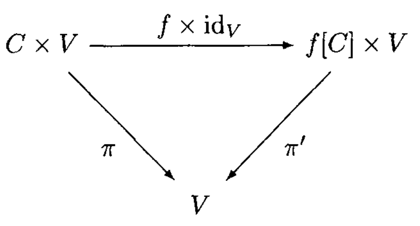
LEMMA 2.14:

(a) *Let V, C be varieties, V arbitrary and C complete. If $f : C \to V$ is a morphism, then $f[C]$ is complete and closed in V.*

(b) *Any closed subset of a complete variety is itself complete.*

(c) *A finite product of complete varieties is complete.*

Proof: (a) The following diagram is commutative, thus for any closed set $F \subseteq f[C] \times V$, the set $\pi'[F] = \pi[(f \times \mathrm{id}_V)^{-1}[F]]$ is closed.



It follows that $C' = f[C]$ is complete. Let $\pi : C' \times \overline{C'} \to \overline{C'}$ be the projection. $\Delta(C') = \Delta(\overline{C'}) \cap (C' \times \overline{C'})$ is closed in $C' \times \overline{C'}$, hence $C' = \pi[\Delta(C')]$ is closed, too.

(b) Let C be complete and F a closed subset. Any closed subset of $F \times G^n$ is closed in $C \times G^n$, hence its image under the projection onto G^n is closed.

(c) If C_1, C_2 are complete and V a variety, then $\pi : C_1 \times C_2 \times V \to V$ factors in $C_1 \times (C_2 \times V) \longrightarrow C_2 \times V \longrightarrow V$, hence is a closed map. ∎

3. Subgroups and quotient groups

3.1 SOME BASIC PROPERTIES. By Proposition 2.4, left and right multiplication with a fixed element as well as conjugation by a fixed element g are isomorphisms. Conjugation $(h, g) \mapsto h^g$ is a morphism.

The Cartesian product of two irreducible sets X, Y is irreducible, hence $X \cdot Y = \mu[X \times Y]$ is irreducible, too.

PROPOSITION 3.1: *A definable subgroup H is closed. Its irreducible components are the cosets of the connected component H°. (This generalizes [Hu] 7.3 and 7.4.)*

Proof: By continuity of multiplication and inverse, $\overline{H} \cdot \overline{H} \subseteq \overline{H \cdot H} = \overline{H}$ and $\overline{H}^{-1} = \overline{H^{-1}} = \overline{H}$, hence $\overline{H}$ is a subgroup. If $\overline{H} \setminus H$ is non-empty, then it a union of cosets of H whence $\dim(\overline{H} \setminus H) \geq \dim H = \dim \overline{H}$: contradiction.

To prove the second assertion, it is enough to show that connected subgroups are irreducible (the converse is obvious). Let H be connected. There is an irreducible component of H, say F, which is generic in H (in the model theoretic sense, see [P2] section 2a). Any intersection $F \cap hF$ for $h \in H$ is still generic, which means $\dim(F \cap hF) = \dim F$, whence $F \cap hF = F$. But H is covered by finitely many translates of F, hence $H = F$ is irreducible. ∎

In particular, a Zariski group is a connected group by Axiom A1.

3.2 THE VARIETY STRUCTURE ON SUBGROUPS AND QUOTIENTS. Let H be a definable subgroup of a Zariski group G. The two variety structures on G/H, left and right coset space, are well behaved varieties in the sense of the following theorem. Because they are isomorphic under inversion, it suffices to consider left cosets.

Notations: Let $p : G \to G/H$ be the canonical surjection and $p^n := (p, \dots, p) : G^n \to (G/H)^n$. If $X \subseteq (G/H)^k \times G^n$, denote $(p^k \times \mathrm{id}_{G^n})^{-1}[X]$ by $\widetilde{X}$.

THEOREM 3.2:

(a) *The natural surjection $\tilde{p} = p^k \times \mathrm{id}_{G^n} : G^{k+n} \to (G/H)^k \times G^n$ maps constructible sets onto constructible ones.*

(b) *If $R \subseteq (G/H)^k \times G^n$ is constructible, then $\dim R = \dim \tilde{p}^{-1}[R] - k \cdot \dim H$. In particular, $\dim G = \dim G/H + \dim H$.*

(c) *Dimension in $(G/H)^k \times G^n$ equals Morley rank.*

(d) *Definability and additivity of dimension hold in $(G/H)^k \times G^n$.*

Proof: First note that G/H is really a variety: the corresponding equivalence relation E_H is the inverse image of H under the morphism $(g, h) \mapsto g^{-1}h$, hence closed.

Note that $(G/H)^k \times G^n = G^{k+n}/(H^k \times \{e\}^n)$. Replacing G^{k+n} by G and $H^k \times \{e\}^n$ by H, one may assume w.l.o.g. $k = 1$ and $n = 0$.

LEMMA 3.3: *If $X \subseteq G/H$ is irreducible and constructible, then the irreducible components of $\widetilde{X}$ are H°-saturated and isomorphic under translations.*

Proof: Let X_i $(i \in I)$ be the irreducible components of $\widetilde{X}$. Then $X_i \subseteq X_i H^\circ \subseteq \overline{X_i H^\circ}$ which is closed irreducible, hence contained in an irreducible component of $\widetilde{X}$, necessarily X_i. So $X_i = X_i H^\circ$ is H°-saturated.

Let $\{h_0, \dots, h_l\} \subseteq H$ be a complete system of representatives of H/H°. Let $Y_i := X_i h_0 \cup \cdots \cup X_i h_l = X_i H^\circ h_0 \cup \cdots \cup X_i H^\circ h_l = X_i(H^\circ h_o \cup \cdots \cup H^\circ h_l) = X_i H$.

It is straightforward to verify that $X_i = \widetilde{X} \cap \overline{X_i}$ implies $Y_i = \widetilde{X} \cap \overline{Y_i}$. Hence Y_i is a relatively closed H-saturated subset of $\widetilde{X}$, i.e. $p[Y_i]$ is a closed subset in X. By irreducibility of X, there is i_0 such that $p[Y_{i_0}] = X$, which means $\widetilde{X} = Y_{i_0} = X_{i_0}h_0 \cup \cdots \cup X_{i_0}h_l$ is the decomposition into irreducible components. ∎

Note that the H-saturation of a constructible set Q is constructible because it equals QH. The closure of an H-saturated set Q is still H-saturated: $\overline{Q} = \overline{QH} \supseteq \overline{Q} \cdot \overline{H} = \overline{Q}H$. In other words, $\overline{p^{-1}[Q]} = p^{-1}[\overline{Q}]$ or $\widetilde{\overline{X}} = \overline{\widetilde{X}}$.

(a) Let $Q \subseteq G$ be constructible. Then $p[Q] = p[QH] = p[\overline{QH}] \setminus p[\overline{QH} \setminus QH]$ is constructible by induction on $\dim QH$, because $p[\overline{QH}]$ is closed (by definition of the topology on G/H) and $p[\overline{QH} \setminus QH]$ is constructible by induction.

(b) Let $R \subseteq G/H$ be constructible and irreducible. Let $F_0 \subset \cdots \subset F_k$ be a chain of relatively closed irreducible subsets of R. Then $\widetilde{F_0} \subset \cdots \subset \widetilde{F_k} \subseteq \widetilde{R}$.

Claim: $\dim \widetilde{F_{i+1}} > \dim \widetilde{F_i}$

Proof: $\overline{\widetilde{F_{i+1}} \setminus \widetilde{F_i}}$ is H-saturated, hence $X = p[\overline{\widetilde{F_{i+1}} \setminus \widetilde{F_i}}]$ is closed. Irreducibility of $F_{i+1} \subseteq F_i \cup X$ implies $X = \overline{F_{i+1}}$. It follows that $\widetilde{F_{i+1}} \setminus \widetilde{F_i}$ is dense in $\widetilde{F_{i+1}}$, whence $\dim \widetilde{F_i} < \dim \widetilde{F_{i+1}}$. ∎

This yields immediately $\dim \widetilde{R} \geq \dim R + \dim \widetilde{F_0} = \dim R + \dim H$.

To show the other inequality, I need some more notations:

Remember that $E_H = \{(g,h) | g^{-1}h \in H\}$ is closed in G^2. Let $\pi_1, \pi_2 : G^2 \to G$ be the two projections and let $\widehat{Q} := \pi_1^{-1}[Q] \cap E_H$. Then the H-saturation of Q equals $\pi_2[\widehat{Q}]$.

If Q is constructible irreducible, define the H-generic dimension of Q, H-gdim Q, to be the π_2-generic fiber dimension of $\widehat{Q}$. Although QH is not irreducible, this is well defined because all components are isomorphic under translations (Lemma 3.3). Moreover, additivity applied to $\widehat{Q}$ (component by component) yields $\dim Q + \dim H = \dim \widehat{Q} = \dim QH + H\text{-gdim}\, Q$ $(*)$.

The H-generic dimension of Q is a precise definition of the generic dimension of the non-empty intersections of Q with H-cosets. The union U of all those intersections $Q \cap gH$ whose dimension differs from H-gdim Q is definable ($= \pi_1[E_H \cap \pi_2^{-1}[\pi_2[\widehat{Q}] \setminus \pi_2[\widehat{Q}, H\text{-gdim}\, Q]]]$). Moreover $Q \not\subseteq \overline{U}$, which means that the H-generic dimension is really generic.

Because dimension equals Morley rank in Zariski groups, Lemma 1.6 allows one to choose relatively closed irreducible subsets Q' of Q such that $Q' \setminus U$ is dense in Q, which implies $H\text{-gdim}\, Q' \leq H\text{-gdim}\, Q$. Inductively one gets a chain

$F_0 \subset \cdots \subset F_{\dim Q} = Q$ of relatively closed irreducible subset of Q such that H-gdim $F_i \leq H$-gdim F_{i+1} for all i.

Let $R_i := \overline{p[F_i]}$ and consider the chain of irreducible sets $R_0 \subseteq \cdots \subseteq R_{\dim Q}$. Then

$$\begin{aligned} R_i = R_{i+1} &\iff \overline{F_i H} = \widetilde{R_i} = \widetilde{R_{i+1}} = \overline{F_{i+1} H} \\ &\iff \text{(by Lemma 3.3)} \ \dim F_i H = \dim \overline{F_i H} = \dim \overline{F_{i+1} H} \\ &\qquad\qquad\qquad\qquad\qquad\qquad = \dim F_{i+1} H \\ &\iff \text{(by } *) \ H\text{-gdim}\, F_i + 1 = H\text{-gdim}\, F_{i+1}. \end{aligned}$$

This last equality is at most $\dim H$ times possible, hence the length of the chain of R_i is at least $\dim Q - \dim H$, whence $\dim p[Q] \geq \dim Q - \dim H$.

(c) Let $R \subseteq G/H$ be constructible. By part (b), $\dim R = \dim \widetilde{R} - \dim H = \dim \overline{\widetilde{R}} - \dim H = \dim \widetilde{\overline{R}} - \dim H = \dim \overline{R}$ and (c) follows from Lemma 1.6.

(d) G/H is still an $\aleph_0$-stable group of finite rank, hence Morley rank (= dimension) is definable and additive. ∎

Now let $H_1 \leq H_2 \leq G^m$ and $K_1 \leq K_2 \leq G^l$ be definable subgroups. It is easy to see that the topology on $H_2/H_1 \times G^n$ is the induced topology of $G/H_1 \times G^n$. Hence the properties of Theorem 3.2 apply also to the variety $H_2/H_1 \times K_2/K_1$.

COROLLARY 3.4:

(a) *The projection $H_2/H_1 \times K_2/K_1 \to H_2/H_1$ maps constructible sets onto constructible sets.*

(b) *Dimension is definable and additive in $H_2/H_1 \times K_2/K_1$.*

(c) *Morley rank equals dimension in $H_2/H_1 \times K_2/K_1$.*

3.3 SMOOTHNESS OF SUBGROUPS AND QUOTIENTS.

PROPOSITION 3.5: *Let G be a smooth Zariski group. If H is a definable subgroup, then G/H is a smooth variety.*

Proof: Let F_1, F_2 be closed irreducible subsets in $G/H \times G^n$. One may assume $n = 0$ (consider $G^{n+1}/H \times \{e\}^n$). According to Lemma 3.3, all irreducible components of $\widetilde{X}$ for irreducible $X \subseteq G/H$ are isomorphic, hence have the same dimension $\dim X + \dim H$.

Let Y be an irreducible component of $F_1 \cap F_2$ and $\widetilde{Y}'$ any irreducible component of $\widetilde{Y}$. Then there are irreducible components $\widetilde{F_i}'$ of $\widetilde{F_i}$ such that $\widetilde{Y}'$ is an component of $\widetilde{F_1}' \cap \widetilde{F_2}'$.

Then $\dim Y = \dim \widetilde{Y}' - \dim H \geq \dim \widetilde{F_1}' + \dim \widetilde{F_2}' - \dim G - \dim H = \dim \widetilde{F_1} - \dim H + \dim \widetilde{F_2} - \dim H - (\dim G - \dim H) = \dim F_1 + \dim F_2 - \dim G/H$. ∎

PROPOSITION 3.6: *Let G be a smooth and rich Zariski group. Then any definable subgroup N provides a smooth variety.*

CLAIM: *There is a closed irreducible set S such that $S \cap gN$ is finite and non-empty for all g in a dense open subset of G:*

Proof: Replacing N by its connected component, one may suppose N to be connected. Richness allows one to find $\dim N$ hypersurfaces H_i of G such that $N \cap H_1 \cap \cdots \cap H_{\dim N}$ is finite and non-empty. Let S be an irreducible component of $H_1 \cap \cdots \cap H_{\dim N}$. By smoothness,

$$\dim S \geq \dim N \cdot (\dim G - 1) - (\dim N - 1) \cdot \dim G = \dim G - \dim N.$$

Let $\widetilde{S} := \{(s,n) | s \in S, n^{-1}s \in N\}$ and let π_i be the projection on the ith coordinate. The π_1-fibers of $\widetilde{S}$ are homeomorphic to N and $\pi_1[\widetilde{S}] = S$, hence $\dim \widetilde{S} = \dim S + \dim N \geq \dim G$. On the other hand, the π_2-fibers are homeomorphic to the intersections $S \cap gN$, and by construction, there is a finite π_2-fiber. But dimension is semi-continuous (Proposition 2.9), hence almost all π_2-fibers are finite. Thus $\dim \widetilde{S} = \dim \pi_2[\widetilde{S}] + \pi_2\text{-gdim}\, \widetilde{S} \leq \dim G + 0$, whence $\dim \widetilde{S} = \dim G$ and $S \cap gN$ is non-empty for almost all $g \in G$. ∎

Now let F_1, F_2 be two closed irreducible subsets of $N \times G^n$ and let Δ be the diagonal in $G^{n+1} \times G^{n+1}$. Because $F_1 \cap F_2$ is homeomorphic to $(F_1 \times F_2) \cap \Delta$, it suffices to show that $\dim X \geq \dim F - \dim(N \times G^n)$ for every closed irreducible $F \subseteq G^{2n+2}$ and each irreducible component X of $F \cap \Delta$.

Let S be as above. Multiplying by an appropriate element, one may suppose $e \in S$. Say $S \cap N = \{e = s_0, s_1, \ldots, s_k\}$. Let $\widetilde{\Delta} := (S \times \{e\}^{2n+1}) \cdot \Delta = \{(s \cdot g_0, g_1, \ldots, g_n; g_0, \ldots, g_n) | s \in S, g_i \in G\}$. Then $\widetilde{\Delta}$ is closed irreducible of dimension $\dim S + \dim(G^{n+1}) = (n+2)\dim G - \dim N$.

Let $s_i\Delta$ be a shorthand for $(s_i, e, \ldots, e) \cdot \Delta$. Because the sets $s_i\Delta$ $(i = 0, \ldots, k)$ are disjoint and

$$F \cap \widetilde{\Delta} = F \cap (s_0\Delta \cup \cdots \cup s_k\Delta) = \underbrace{(F \cap s_0\Delta)}_{=F\cap\Delta} \cup \cdots \cup (F \cap s_k\Delta),$$

each irreducible component of $F \cap \Delta$ is also a component of $F \cap \widetilde{\Delta}$. Then smoothness of G implies for each irreducible component Y of $F \cap \widetilde{\Delta}$ that $\dim Y \geq \dim F + \dim \widetilde{\Delta} - \dim G^{2n+2} = \dim F - \dim(N \times G^n)$. ∎

COROLLARY 3.7: *Let G be a rich and smooth Zariski group. Then each variety obtained by taking definable subgroups and quotients by definable subgroups is smooth.*

PROPOSITION 3.8: *If G is a Zariski group, H a definable connected subgroup and N a definable normal subgroup of H, then H/N is itself a Zariski group. It is smooth if G is smooth and either $H = G$ or G is rich.*

Proof: H/N is a variety, hence $(H/N)^n$ is also a variety and thus there is a finite-dimensional Noetherian topology on $(H/N)^n$. H being connected, H/N is connected, too, thus irreducible (3.1). The diagonal is closed by definition of varieties. Quantifier elimination comes from Corollary 3.4 (a). Lemma 2.5 implies immediately that the compatibility maps of Axiom A3, the inverse map and the multiplication are morphisms.

To get a countable language, it suffices to consider the restrictions of the basic closed sets of G. The chain condition of Axiom B2 is inherited from G.

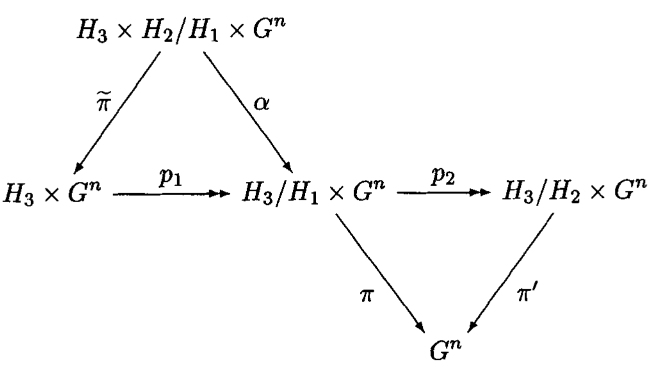
Finally, Morley rank equals dimension by Corollary 3.4 (c) and is obviously finite.

The smoothness conditions are clear from the preceding propositions. ∎

3.4 COMPLETE GROUPS.

PROPOSITION 3.9: *Let $H_1 \leq H_2 \leq H_3$ be varieties. Then H_3/H_1 is complete iff H_3/H_2 and H_2/H_1 are complete.*

Proof: Assume first that H_2/H_1 and H_3/H_2 are complete. Let π and π' be the projections and p_1, p_2 the canonical surjections as indicated in the following diagram:



Let $F \subseteq H_3/H_1 \times G^n$ be a closed set and put $\widetilde{F} := (p_1^{-1} \circ p_2^{-1} \circ p_2)[F]$. This is the H_2-saturation of the inverse image of F in $H_3 \times G^n$. It is sufficient to show that $\widetilde{F}$ is closed, since then $p_2[F] = (p_2 \circ p_1)[\widetilde{F}]$ is closed and hence $\pi[F] = \pi'[p_2[F]]$ is closed by completeness of H_3/H_2.

Let $\alpha : H_3 \times H_2/H_1 \times G^n \to H_3/H_1 \times G^n$ be the morphism $(g, hH_1, \bar{z}) \mapsto (ghH_1, \bar{z})$ and let $\tilde{\pi} : H_3 \times H_2/H_1 \times G^n \to H_3 \times G^n$ be the projection, which is a closed map by completeness of H_2/H_1 (Lemma 2.13). Then

$$\widetilde{F} = \{(g, \bar{z}) | \exists hH_1 \in H_2/H_1 \text{ such that } (ghH_1, \bar{z}) \in F\} = (\tilde{\pi} \circ \alpha^{-1})[F]$$

is closed by completeness of H_2/H_1 and continuity of α.

Now assume that H_3/H_1 is complete. Then H_2/H_1 is a closed subset and H_3/H_2 is isomorphic to the quotient $(H_3/H_1)/(H_2/H_1)$, hence both are complete by 2.14 (b) and (a). ∎

THEOREM 3.10:

(a) *Let C be an irreducible complete subset in a Zariski group G such that $e \in C$. Then the (normal) subgroup generated by C is definable and complete.*

(b) *G contains a maximal complete connected subgroup G^c. This subgroup is unique and normal in G and G/G^c contains no infinite complete subgroup.*

Proof: (a) An irreducible sets is obviously indecomposable. Then by Zil'ber's indecomposability theorem, the normal subgroup generated by C is definable and equals $C^{g_1} \cdot \ldots \cdot C^{g_k}$ for some $g_1, \ldots, g_k \in G$. This is the morphical image of the complete set $C^{g_1} \times \cdots \times C^{g_k}$, hence complete. The subgroup generated by C is a closed subset of the normal subgroup generated by C, thus complete, too.

(b) Let G^c be the subgroup generated by all irreducible complete subsets containing e. Again by Zil'ber's theorem, it equals the product of a finite number of them, hence is complete. It is obviously a maximal connected complete subgroup, normal and unique.

Let G^{cc} be such that $G^{cc}/G^c = (G/G^c)^c$. Then G^{cc} is connected by definition and complete by Proposition 3.9, hence $G^{cc} \leq G^c$. ∎

THEOREM 3.11: *Assume dimension to be semi-continuous. Then G^c is central in G.*

Remark: This theorem generalizes the corollary of theorem 14 of [R]. In fact, it is possible to prove it without the semi-continuity property (i.e. in a not necessarily smooth Zariski group), but this requires more model theory (see [J] theorem 5.33).

Proof: Consider the morphism $\zeta : G^c \times G \to G^c$, $(c, g) \mapsto [c, g]$. Its graph $\Gamma_\zeta \subseteq G^c \times G \times G^c$ is closed and irreducible, hence by completeness of G^c, the set $T = \{(g, h) | \exists c \in G^c, h = [c, g]\}$, which is the projection of Γ_ζ on $G \times G^c$, is also closed irreducible. Let $\pi : G \times G^c \to G$ be the projection; then the π-fiber over e is reduced to $\{e\}$. By the hypothesis, π-fibers of T are generically finite.

Thus the sets $\{[c, g_i] | c \in G^c\}$ are finite for generic g_1, g_2. Now $[c, g_1 g_2] = [c, g_2] \cdot [c, g_1]^{g_2}$, hence the set $\{[c, g_1 g_2] | c \in G^c\}$ is also finite. But the product of two dense constructible sets is the whole of G, which shows that all the π-fibers of T are finite. (Cf. [P2] 2.4: a dense constructible subset of an irreducible set is generic in Poizat's sense).

Fix $g \in G$ and suppose $C := \{[c, g] | c \in G^c\} = \{c_1, \ldots, c_k\}$. Define a morphism $\chi : G^c \to C$, $c \mapsto [c, g]$. Then the sets $\chi^{-1}(c_i)$ form a finite partition of the connected subgroup G^c into closed sets, forcing $k = 1$, $c_1 = e$ and $g \in \mathbb{C}_G(G^c)$. It follows that $G^c \subseteq \mathbb{Z}(G)$. ∎

COROLLARY 3.12: *A simple smooth Zariski group does not contain any infinite complete subset.*

Proof: If C is infinite complete and $c \in C$, then $c^{-1}C$ in still complete and contains e. By simplicity, G equals the normal subgroup generated by $c^{-1}C$, which is complete by Theorem 3.10. A simple smooth Zariski group is rich by Lemma 2.8. As dimension is semi-continuous by Proposition 2.9, the preceding theorem applies and shows that G is Abelian: contradiction. ∎

3.5 PARABOLIC GROUPS.

Definition 3.13: A definable subgroup H of a Zariski group G is **parabolic** if G/H is a complete variety.

A subgroup H gives rise to the left and to the right coset space which are isomorphic under inversion. Thus the notion of a parabolic subgroup is unambiguous: H is left parabolic iff it is right parabolic.

The following proposition is just a reformulation of Proposition 3.9 (note that finite sets are always complete):

PROPOSITION 3.14: *Let $P_1 \leq P_2 \leq P_3$ be definable subgroups. Then P_1 is parabolic in P_3 iff P_1 is parabolic in P_2 and P_2 in P_3. In particular, P_2 is parabolic in P_3 iff P_2^o is parabolic in P_3.*

COROLLARY 3.15: *Every parabolic subgroup of G contains a minimal parabolic subgroup. Minimal parabolic subgroups are connected and do not contain proper parabolic subgroups.*

PROPOSITION 3.16: *Let G be a smooth and rich Zariski group. If P_1, P_2 are parabolic subgroups such that P_2 normalizes P_1, then $P_1 \cap P_2$ is a parabolic subgroup.*

Proof: P_1P_2 is a subgroup of G because P_2 normalizes P_1, in particular it is closed in G. Hence $(P_1P_2)/P_1$ is complete as a closed subset of G/P_1.

There is a natural bijection $\beta : P_2/(P_1 \cap P_2) \to (P_1P_2)/P_1$ coming from the inclusion $P_2 \hookrightarrow P_1P_2$. By Lemma 2.5, β is a morphism.

Because $P_2/(P_1 \cap P_2) \times (P_1P_2)/P_1$ is a smooth variety (Corollary 3.7), β is an isomorphism by Corollary 2.11, thus $P_1 \cap P_2$ is parabolic in P_2 and hence in G by Proposition 3.14. ∎

PROPOSITION 3.17: *Let G be a smooth and rich Zariski group. All minimal parabolic subgroups are conjugate.*

Proof: Let P and Q be two minimal parabolic subgroups. P acts on the complete variety G/Q by left multiplication. Let Y be an orbit of this action. Then $\overline{Y}$ and $\overline{Y} \setminus Y$ are stable under the action of P, hence both are unions of orbits. Thus $\dim(\overline{Y} \setminus Y) < \dim \overline{Y}$ implies that orbits of minimal dimension are closed. In particular, closed orbits exist always (this is exactly as in algebraic geometry, see [Hu] section 8).

Now let $g \in P$ be such that the orbit $Y := P \cdot gQ$ is closed, hence complete as subset of the complete variety G/Q. Let P_{gQ} be the stabilizer of gQ, i.e. the group $\{p \in P | p \cdot gQ = gQ\}$.

The action provides a bijective morphism $\beta : P/P_{gQ} \to Y$. The graph being a subset of $P/P_{gQ} \times G/Q$ which is smooth by 3.7, Corollary 2.11 shows that β is an isomorphism.

Thus P/P_{gQ} is complete. But P is minimal parabolic and has no proper parabolic subgroups, forcing $P = P_{gQ}$. This means $p \cdot gQ \subseteq gQ$ for all $p \in P$, or $P^g \subseteq Q$. Now P^g is a parabolic subgroup of G because conjugation is an isomorphism. Minimality of Q then proves $P^g = Q$. ∎

Remark: In fact, it is possible to show the preceding two propositions without the richness condition, using the smoothness of G only and a more elaborate version of Corollary 2.11.

THEOREM 3.18: *A rich and smooth Zariski group contains a unique minimal normal parabolic subgroup G^{a}. This subgroup is connected and $G^{\mathrm{aa}} = G^{\mathrm{a}}$.*

Proof: Let G^{a} be the intersection of all normal parabolic subgroups. This is again a normal definable subgroup, and parabolic by Proposition 3.16. Unicity and minimality are obvious.

$(G^{\mathrm{a}})^{\mathrm{o}}$ is still a normal parabolic subgroup, hence G^{a} is connected. Finally, G^{a} must not have any proper normal parabolic subgroups due to 3.14. ∎

Thus a smooth and rich Zariski group G has two normal subgroups G^{a} and G^{c} (Theorems 3.10 and 3.18) whose properties are resumed in the following table:

$G^{\mathrm{aa}} = G^{\mathrm{a}}$	$(G/G^{\mathrm{c}})^{\mathrm{c}} = G^{\mathrm{c}}/G^{\mathrm{c}}$
$G^{\mathrm{cc}} = G^{\mathrm{c}}$	$(G/G^{\mathrm{a}})^{\mathrm{a}} = G^{\mathrm{a}}/G^{\mathrm{a}}$
$G^{\mathrm{ca}} = \{e\}$	$(G/G^{\mathrm{a}})^{\mathrm{c}} = G/G^{\mathrm{a}}$
$G^{\mathrm{ac}} = G^{\mathrm{c}} \cap G^{\mathrm{a}}$	$(G/G^{\mathrm{c}})^{\mathrm{a}} = G^{\mathrm{a}}G^{\mathrm{c}}/G^{\mathrm{c}}$

Furthermore, $G^{\mathrm{c}} \leq \mathbb{Z}(G)$ and obviously $G^{\mathrm{c}} = G \iff G^{\mathrm{a}} = \{e\}$.

The "dual" equivalence $G^{\mathrm{a}} = G \iff G^{\mathrm{c}} = \{e\}$ is false even for algebraic groups. But if G is an algebraic group, $G^{\mathrm{a}} = G$ is equivalent to the fact that G is affine (by a theorem of Chevalley, see [R] theorem 16) and implies that $G^{\mathrm{c}} = \{e\}$. Thus the property $G^{\mathrm{a}} = G$ might be a possible definition of affine Zariski groups, but it is not clear whether this implies $G^{\mathrm{c}} = \{e\}$ and whether a closed subgroup of an affine group is still affine.

A dual version of the first property could be Rosenlicht's theorem (true for algebraic groups, see [R] theorem 13, but open for Zariski groups), namely $(G/\mathbb{Z}(G))^{\mathrm{a}} = G/\mathbb{Z}(G)$.

4. There are no bad Zariski groups

A **Borel subgroup** is a maximal connected solvable definable subgroup. In algebraic groups, the Borel subgroups are exactly the minimal parabolic subgroups. Most of the theory of affine algebraic groups is based on the structure of the Borels. While it was possible to show some of their properties for the minimal parabolic subgroups (for example, Proposition 3.17), it is in general not known whether Borels are parabolic or not. If Cherlin's conjecture holds, then Borels of simple Zariski groups have to be parabolic. However, in special situations, parabolic subgroups can be constructed, much in a way as projective space (a complete variety, too) is constructed.

THEOREM 4.1: *Let G be a smooth Zariski group with semi-continuous dimension. Suppose H is a connected definable subgroup such that $H \cap H^g = \{e\}$ for all $g \in G \setminus H$. Then H is a parabolic subgroup of G.*

Proof: First note that the hypothesis implies $H = \mathbb{N}_G(H)$ and that being in the same conjugate of H defines an equivalence relation E on $\mathbb{H} := \bigcup_{g\in G} H^g \setminus \{e\}$. Thus conjugation $H \mapsto H^g$ induces a map $\alpha : G/\mathbb{N}_G(H) \to \mathbb{H}/E$. It is quite easy to show that $\mathbb{H}/E$ is complete, so the underlying idea of the proof is to show that α is an isomorphism of varieties. But I don't know whether $\mathbb{H}/E$ is a well-behaved variety at all, so I have to do this construction implicitly.

Let $\Gamma_H := \{(g, h^g) | h \in H, g \in G\} = \bigcup_{g\in G} (Hg \times H^g)$. Note that Γ_H is the closure of the $(E \times H)$-saturation of Γ_α. Let $\Gamma_{H,n} := \Gamma_H \times \Delta(G)^n$. Then $\Gamma_{H,n}$ is a closed irreducible subset of G^{2n+2} of dimension $\dim H + (n+1)\dim G$. Let π_1 (resp. π_2) : $G^{2n+2} \to G^{n+1}$ be the projection on the coordinates with odd (even) index. The π_1-fiber over $(x, \bar{g})$ equals $H^x \times \{\bar{g}\}$; and the π_2-fiber over some point $(h^x, \bar{g})$ equals $\mathbb{N}_G(H)x \times \{\bar{g}\} = Hx \times \{\bar{g}\}$, except over $(e, \bar{g})$ where it is $G \times \{\bar{g}\}$. Hence π_1-gdim $\Gamma_{H,n} = \pi_2$-gdim $\Gamma_{H,n} = \dim H$.

To simplify notations, fix n and let $\Gamma = \Gamma_{H,n}$.

Left coset space G/H and right coset space $H\backslash G$ are isomorphic, thus one is complete iff the other is. To simplify notations, let us consider the right coset space $H\backslash G = \{Hg | g \in G\}$. To every set $X \subseteq H\backslash G \times G^n$ associate the set $\gamma(X) := \pi_2[\Gamma \cap \pi_1^{-1}[\widetilde{X}]] = \{(h^g, g_1, \dots, g_n) | h \in H, (Hg, g_1, \dots, g_n) \in X\} \subseteq G \times G^n$. In some sense, γ is a map whose graph is Γ.

CLAIM: *$\gamma(X)$ is irreducible if X is.*

Proof: Let $p : G \times G^n \to H\backslash G \times G^n$ be the canonical surjection. By Lemma 3.3, $\widetilde{X} = p^{-1}[X]$ is irreducible. Let $\zeta : (h, g) \mapsto h^g$ be the conjugation. Now $\gamma(X) = (\zeta \times \mathrm{id}_{G^n})[H \times \widetilde{X}]$ is irreducible as the continuous image of an irreducible set. ∎

CLAIM: *$\gamma(X)$ is closed if X is closed irreducible.*

Proof: The proof is analogous to that of Proposition 2.10. Let X be as above and closed irreducible, $Y := \overline{\gamma(X)}$ and $Y' := \overline{Y \setminus \gamma(X)}$. Then $\dim Y' < \dim Y$. According to the hypothesis of the theorem

$$\pi_2^{-1}[\gamma(X)] \cap \Gamma \subseteq \left(\pi_1^{-1}[X] \cap \Gamma\right) \cup \left(\{e\} \times G^{2n+1}\right).$$

whence

$$(*) \qquad \pi_2^{-1}[Y] \cap \Gamma \subseteq \left(\pi_1^{-1}[X] \cap \Gamma\right) \cup \left(\pi_2^{-1}[Y'] \cap \Gamma\right) \cup \left(\{e\} \times G^{2n+1}\right).$$

All these sets are closed, hence any irreducible component of the left side is contained in an irreducible component of one of the terms on the right side.

G^{2n+2} being smooth, for each irreducible component I of $\pi_2^{-1}[Y] \cap \Gamma$,

(**)

$$\begin{aligned}\dim I &\geq \dim \pi_2^{-1}[Y] + \dim \Gamma - \dim G^{2n+2} \\ &= \dim Y + (n+1)\cdot \dim G + \dim H + (n+1)\cdot \dim G - (2n+2)\cdot \dim G \\ &= \dim Y + \dim H.\end{aligned}$$

Let $F_1, \ldots, F_m$ be the irreducible components of Y'. In particular $\dim F_i \leq \dim Y' < \dim Y$ and $\pi_2^{-1}[Y'] \cap \Gamma = \left(\pi_2^{-1}[F_1] \cap \Gamma\right) \cup \cdots \cup \left(\pi_2^{-1}[F_m] \cap \Gamma\right)$. Now by (*) any irreducible component of $\pi_2^{-1}[F_i] \cap \Gamma$ is either an irreducible component of $\pi_2^{-1}[Y] \cap \Gamma$ or contained in $(\pi_1^{-1}[X] \cap \Gamma) \cup (\{e\} \times G^{2n+1})$. For a given index i, either $F_i \subseteq \{e\} \times G^n$ or $F_i \setminus (\{e\} \times G^n)$ is dense in F_i. In the second case, the π_2-generic fiber dimension of Γ over F_i is $\dim H$, hence by additivity

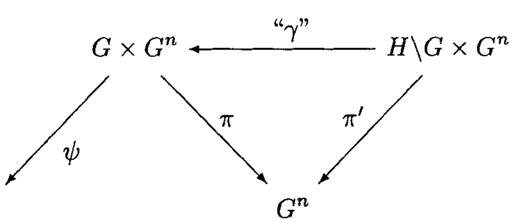
$$\dim\left(\pi_2^{-1}[F_i] \cap \Gamma\right) = \dim F_i + \dim H < \dim Y + \dim H$$

which contradicts (**). It follows that $Y' \subseteq \{e\} \times G^n$ and hence

$$\pi_2^{-1}[Y] \cap \Gamma \subseteq \left(\pi_1^{-1}[X] \cap \Gamma\right) \cup \left(\{e\} \times G^{2n+1}\right).$$

Now consider $Y \subseteq G \times G^n$ and let $\pi : G \times G^n \to G^n$ be the projection. Y is a closed irreducible set whose π-generic fiber dimension is at least $\dim H$. Any component F_i included in $\{e\} \times G^n$ would provide a finite π-fiber contradicting the semi-continuity of the dimension. It follows that $\gamma(X)$ is closed. ∎

To finish the proof of the theorem, start with a closed irreducible $X \subseteq H\backslash G \times G^n$. Let $\pi' : H\backslash G \times G^n \to G^n$ and $\psi : G \times G^n \to G$ be the projections.



Obviously, $\pi'[X] = \pi[\gamma(X)] = \psi^{-1}(e) \cap \gamma(X)$ which is a closed set. So H is a parabolic subgroup. ∎

Recall that an $\aleph_0$-stable group of finite rank is called **bad** if it is non-solvable, and all proper connected definable subgroups are nilpotent.

PROPOSITION 4.2: *If G is a simple bad smooth Zariski group, then its Borel subgroups are parabolic.*

Proof: In a simple bad group, Borel subgroups are auto-normalizing and two distinct Borels intersect in $\{e\}$, see [P2] theorem 3.31. A simple group is rich by Lemma 2.8. As dimension is semi-continuous (Proposition 2.9), Theorem 4.1 applies. ∎

PROPOSITION 4.3: *There are no bad smooth Zariski groups. (Confer [Hu] 21.4.)*

Proof: If G is a such a group, it has a simple bad quotient group ([P2] theorem 3.31), which is still a smooth Zariski group (Proposition 3.8). By Corollary 3.12, G has no complete infinite subset.

Let B be a Borel. B is nilpotent, hence its center is non-trivial. Take $c \in \mathbb{Z}(B)$ and let $\phi : G \to G$ be the morphism $g \mapsto g^c g^{-1}$. Then ϕ is constant on B-cosets, in particular $\phi[B] = \{e\}$. Thus ϕ factors through G/B providing a morphism $\bar{\phi} : G/B \to G$ (Lemma 2.5).

Now $\bar{\phi}[G/B]$ is irreducible and complete, because G/B is complete, therefore $\bar{\phi}[G/B] = \{e\}$. It follows that $\phi = e$ and $c \in \mathbb{Z}(G)$, that is $\mathbb{Z}(B) \subseteq \mathbb{Z}(G)$, contradicting the simplicity of G. ∎

THEOREM 4.4: *Any smooth, rich, non-nilpotent Zariski group (in particular any simple smooth Zariski group) interprets an algebraically closed field.*

Proof: If the group contains a connected, definable, solvable, non-nilpotent subgroup, it interprets an algebraically closed field by Zil'ber's theorem ([Z], see [P2] corollary 3.20). Otherwise any minimal connected definable non-solvable subgroup is bad. It is smooth by 3.7, hence Proposition 4.3 yields a contradiction. ∎

4.1 OPEN PROBLEMS.

- A non-singular algebraic variety (over an algebraically closed field) satisfies the dimension formula. Because algebraic groups can't have singularities, they are smooth Zariski groups. Question: Are all Zariski groups smooth? (It is known that the dimension formula holds for cosets of definable subgroups, see [J] proposition 5.21.)
- As pointed out by Poizat ([P2] p. 144), a group interpretable in an algebraically closed field is a Zariski group in a canonical way. Is this true for any $\aleph_0$-stable group of finite rank?

- By results of Hrushovski (see [P2] corollary 2.27), a simple group of finite Morley rank is interpretable in any infinite field that is interpretable in the group, provided that the field is endowed with the full structure coming from the group. To solve completely Cherlin's conjecture for Zariski groups, it remains to show that the group is interpretable in the pure field structure. For this, it is sufficient to show that the field is of dimension 1: Because the field is defined on an interpretable subgroup, it is smooth, and hence theorem A of [HZ1] implies that the field is pure.

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